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Exact closed-form solutions for the two-fluid oscillatory flow and Navier's partial slip boundary condition

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Abstract

Two-fluid flows occur when immiscible moving fluids of different properties are in contact. Exact closed- form solutions are presented for oscillatory two-fluid flows. Exact solutions are important not only due for its applicability to specific problems but also serve as accuracy standards for approximate solutions. These oscillatory solutions are governed by geometry, the viscosity ratios, and the normalized frequency *s*. The velocity profiles for large *s* are different from those of small *s*. As $s \to 0$ the solutions may approach a steady-state. For non-zero *s* there is a phase difference between the two fluids. An important result is that Navier's partial slip condition fails for oscillatory flows.

Keywords: *Two-fluid; Oscillatory; Viscous flow; Exact solution; Navier; Slip.*

1. Introduction

Most literature in fluid dynamics are concerned with a single homogeneous fluid. However, in certain instances two immiscible fluids are in contact. For example, in order to reduce the resistance of the flow of a higher viscosity fluid (e.g. oil), a layer of lower viscosity fluid (e.g. water) may be introduced near the container boundary [1]. The same principle applies to the ultra-low resistance of superhydrophobic micro-fluidics [2], where a layer of gas supports the bulk liquid flow. Two-fluid flow is also utilized in extraction processes in chemical engineering [3]. In addition, two-fluid flow is an important model for the flow of particulate fluids, such as blood flow in the microcirculation [4].

The steady, two-fluid channel flow was solved by Bird et al [5]. The steady two-fluid flow in a tube was first proposed by Vand [6] originally for the flow of ink. Hayes [7], Sharan and Popel [8] applied the two-fluid model to blood flow in small vessels.

There are some reports on unsteady two-fluid flows. Kapur and Shukla [9] studied an exponential increasing unsteady channel flow, and Bhattacharyya [10] studied the channel flow due to oscillatory pressure. These two references are limited to the case when the two fluids have exactly the same thickness. The two-fluid oscillatory flow in a channel was studied by Wang [11]. Recently Wang [12], using infinite series, solved the two-fluid starting flow due to a suddenly applied pressure gradient, and Ng [13] considered the two-fluid flow due to an impulsively translated channel boundary. The only two-fluid unsteady flow in a tube was reported by Bugliarello and Sevilla [14] who sketched the oscillatory flow solution.

Oscillatory flow is fundamental in fluid mechanics. Examples include fluid motion due to boundary oscillations, such as that caused by external vibrations. Also pulsatile flow, which is

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necessarily oscillatory, can originate from a reciprocal pump or a mammalian heart.

The purpose of the present paper is to present the closed-form solutions of two-fluid oscillatory flow. Relevant previous solutions shall be discussed. Closed-form solutions only exist for parallel flows and concentric flows. These exact solutions serve as benchmarks for approximate methods, such as numerical, infinite series, or asymptotic results [15]. From these exact solutions, we are able to determine whether Navier's partial slip boundary condition [16] can be applied to unsteady oscillatory flows.

2. Formulation

The assumptions are as follows. We consider the laminar flow of two incompressible immiscible Newtonian fluids. The flow is parallel in the region considered and the entrance and exit effects can be neglected. The interface of the two fluids is stable where velocity and shear stress are matched. No slip occurs on the solid surfaces.

Consider the two-dimensional two-fluid problems shown in Figure1 (a-c). The flow is parallel to the boundary with velocity $u'(t', y')$ where t' is the time and y' is the coordinate normal to the plate at $y'=0$. There is a layer of a different fluid separating the plate and the interior bulk fluid. Let the subscripts 1 and 2 denote the bulk fluid and the boundary fluid respectively. The Navier-Stokes equation for parallel flow reduce to (e.g. [17])

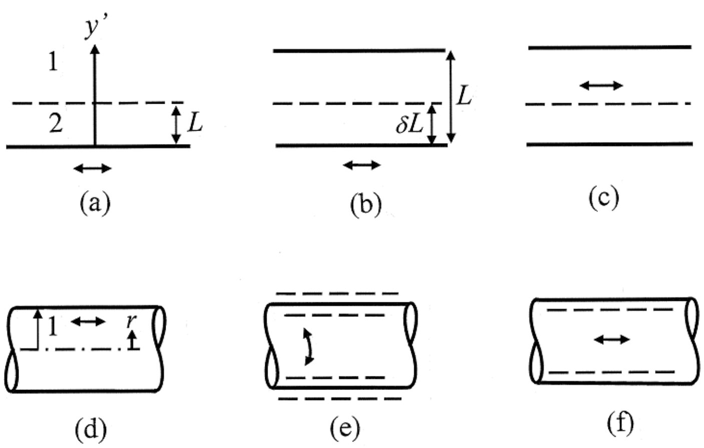


Figure1. (a) Oscillating plate and two fluids (b) channel with bottom wall oscillating (c) channel with an oscillating pressure gradient (d) longitudinal oscillation of a cylinder wall (e) rotational oscillation of a cylinder about its axis (f) tube with oscillating longitudinal pressure gradient. Dashed lines indicate the interface and solid lines the containing boundary.

$$\frac{\partial u_i'}{\partial t'} = -\frac{1}{\rho_i}\frac{\partial p'}{\partial x'} + \nu_i \frac{\partial^2 u_i'}{\partial y'^2} \qquad i=1,2 \tag{1}$$

where ρ is the density, p' is the pressure, and ν is the kinematic viscosity. The primes mean dimensional and the subscripts indicate the fluid region. We assume the lower fluid has higher density. At the interface the velocities and shear stresses match

$$u_1' = u_2' \tag{2}$$

$$\mu_1 \frac{\partial u_1'}{\partial y'} = \mu_2 \frac{\partial u_2'}{\partial y'} \tag{3}$$

If there is no pressure gradient and only the boundary oscillates, say

$$u_2' = U\cos(\omega t') \tag{4}$$

where U is the amplitude and ω is the frequency. Normalize all lengths by a characteristic length L, the time by L^2/ν_1, the velocity by U and drop primes. Eq. (1) becomes

$$u_{1t} = u_{1yy}, \quad \beta^2 u_{2t} = u_{2yy} \tag{5}$$

where the ratio of kinematic viscosity is

$$\beta^2 = \frac{\nu_1}{\nu_2} \tag{6}$$

If an oscillatory pressure gradient of magnitude G_0 causes the motion, then

$$\frac{dp'}{dx'} = -G_0\cos(\omega t') \tag{7}$$

where G_0 is the amplitude. We set $U = G_0 L^2/\mu_1$ and the governing equations are

$$u_{1t} = \cos(s^2 t) + u_{1yy}, \qquad \beta^2 u_{2t} = \alpha^2\cos(s^2 t) + u_{2yy} \tag{8}$$

Here the non-dimensional frequency parameter and the dynamic viscosity ratio are

$$s^2 = \frac{\omega L^2}{\nu_1}, \quad \alpha^2 = \frac{\mu_1}{\mu_2} \tag{9}$$

In either case the matching conditions become

$$u_1 = u_2, \qquad \alpha^2 u_{1y} = u_{2y} \tag{10}$$

In what follows we shall present the basic individual cases.

3. Flows bounded by parallel flat plates

I) In-plane oscillation of a single plate.

Figure1(a) shows an oscillating plate in an infinite fluid separated by a layer of another fluid. Let L, the characteristic length, be the thickness of the layer. The governing equations are Eqs.(5) with the boundary conditions

$$u_1(t,\infty) = 0, \quad u_2(t,0) = \cos(s^2 t) \tag{11}$$

and the matching conditions Eqs.(10) are at y=1. Since the governing equations are linear, let $i = \sqrt{-1}$ and

$$u_1 = e^{is^2 t} f_1(y), \quad u_2 = e^{is^2 t} f_2(y) \tag{12}$$

where only the real part of the product has physical significance. Eqs.(5) become

$$is^2 f_1 = f_1''(y), \qquad is^2\beta^2 f_2 = f_2'' \tag{13}$$

The solution is

$$f_1 = c_1 e^{-k_1 y} \tag{14}$$

$$f_2 = \cosh(k_2 y) + c_2\sinh(k_2 y) \tag{15}$$

where

$$k_1 = \sqrt{i}s, \quad k_2 = \sqrt{i}s\beta, \qquad \sqrt{i} = (1+\sqrt{-1})/2 \tag{16}$$

and

$$c_1 = \frac{\beta e^{k_1}}{\beta\cosh k_2 + \alpha^2\sinh k_2}, \qquad c_2 = -\frac{\beta + \alpha^2\coth k_2}{\alpha^2 + \beta\coth k_2} \tag{17}$$

The exact closed-form solution Eqs. (12,14-17) is valid for arbitrary α, β, s. Figure 2 shows some typical velocity profiles for frequency parameter s=1. For a single fluid, $\alpha = \beta = 1$ and our solution reduces to Stokes' second problem [18].

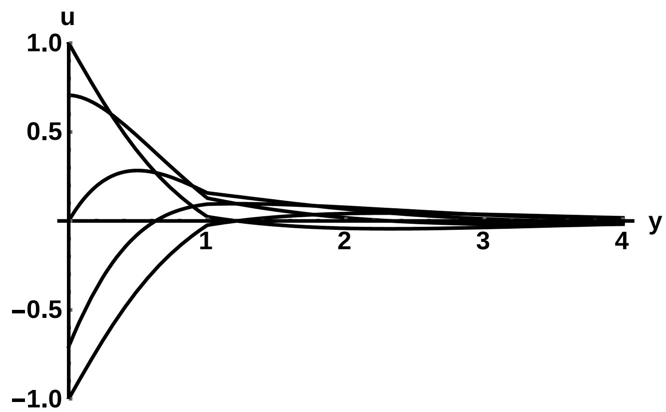


Figure 2. Typical velocity profiles for oscillating plate in infinite fluid (Figure1(a)) $s = 1,\ \alpha = \beta = 2$. From top at left: $ts^2 = 0, 0.25\pi, 0.5\pi, 0.75\pi, \pi$.

II) A channel with one wall oscillating in its own plane.

In this case let L be the channel width, and δL be the thickness of the second fluid (Figure1(b)). Eqs.(13) remain the same but the boundary conditions differ. The solution is

$$f_1 = c_1 \sinh[k_1(y-1)] \tag{18}$$

and f_2 retains the same form as Eq.(15). The coefficients are

$$c_1 = \frac{\beta}{-\beta \cosh k_3 \sinh k_4 - \alpha^2 \cosh k_4 \sinh k_3}$$
$$c_2 = \frac{\alpha^2 \cosh k_4 \coth k_3 + \beta \sinh k_4}{-\beta \sinh k_4 \coth k_3 - \alpha^2 \cosh k_4} \tag{19}$$

Here

$$k_3 = \sqrt{i}\beta\delta s, \quad k_4 = \sqrt{i}(1-\delta)s \tag{20}$$

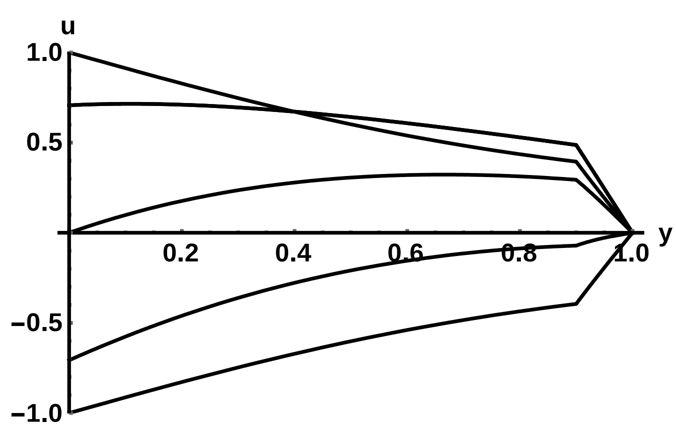


Figure 3. Typical velocity profiles in a channel with one plate oscillating (Figure1(b)) $s = 5, \delta = 0.9, \alpha = \beta = 0.3$. From top at left: $ts^2 = 0, 0.25\pi, 0.5\pi, 0.75\pi, \pi$.

Typical velocity profiles are shown in Figure 3 for a thick lower fluid. In this case a steady state solution exists. The steady state velocity profile is piecewise linear, given as

$$u_1 = \frac{1-y}{1+\alpha^2\delta-\delta}, \quad u_2 = 1-\frac{\alpha^2 y}{1+\alpha^2\delta-\delta} \tag{21}$$

Our oscillatory solution as $s \to 0$ agrees with the steady state solution.

III) A channel with an oscillatory pressure gradient

Here the pressure gradient causes the flow (Figure1 (c)),The length scale L is the width of the channel. Substitute Eqs.(12) into Eqs.(8) to obtain

$$is^2 f_1 = 1 + f_1'', \qquad is^2\beta^2 f_2 = \alpha^2 + f_2'' \tag{22}$$

The boundary conditions are

$$f_1(1) = 0, \quad f_2(0) = 0 \tag{23}$$

$$f_1(\delta) = f_2(\delta), \quad \alpha^2 f_1'(\delta) = f_2'(\delta) \tag{24}$$

After some work, the solution is

$$\begin{aligned} f_1 &= \frac{-i}{s^2}\{1-\cosh[k_1(y-1)]+c_1\sinh[k_1(y-1)]\} \\ f_2 &= \frac{-i\alpha^2}{\beta^2 s^2}\left[1-\cosh(k_2 y)+c_2\sinh(k_2 y)\right] \end{aligned} \tag{25}$$

The constants k_1, k_2, k_3 are defined in Eqs.(16,20) and

$$\begin{aligned} c_1 &= \frac{(\alpha^2-\beta^2+\beta^2\cosh k_3)\cosh k_2-\alpha^2-\alpha^2\beta\sinh k_2\sinh k_3}{\beta(\beta\sinh k_3\cosh k_2-\alpha^2\cosh k_3\sinh k_2)} \\ c_2 &= \frac{\beta^2+(\alpha^2-\beta^2-\alpha^2\cosh k_2)+\beta\sinh k_2\sinh k_3}{\beta\sinh k_3\cosh k_2-\alpha^2\cosh k_3\sinh k_2} \end{aligned} \tag{26}$$

Figure 4 shows typical velocity profiles for s=5. By setting s=0, we can solve for the steady state solution, which is composed of parabolic arcs.

$$\begin{aligned} u_1 &= \frac{-(1-y)^2}{2}+c_1(y-1), \qquad c_1 = -\frac{(1-\delta)^2+\alpha^2\delta(2-\delta)}{2(1-\delta+\alpha^2\delta)} \\ u_2 &= \frac{-\alpha^2 y^2}{2}+c_2 y, \qquad c_2 = \frac{\alpha^2(1-\delta^2+\alpha^2\delta^2)}{(1-\delta+\alpha^2\delta)} \end{aligned} \tag{27}$$

Bhattacharyya [10] considered oscillatory pressure in a channel, but the two fluid layers has exactly the same height, which is seldom true. Wang [11] studied the oscillatory channel flow with similar boundary fluid layers on both walls. This restricts the two fluid densities to be equal or near equal.

4. Flows bounded by a circular cylinder

For two fluid motion concentric about a circular cylinder, the density must be almost equal (so gravity would not distort the interface) or the fluid contains particular matter such that, due to steric hindrance, a particle free layer exists near the solid boundary. For example, in microcirculatory blood flow there is a core containing red blood cells and a boundary layer of mostly cell-free plasma. It has been shown that the two fluid model is well suited to describe experimental phenomena (such as Fahraeus-Linqvist effect) for blood flow in vessels of 40 to

1000 microns in diameter [19].

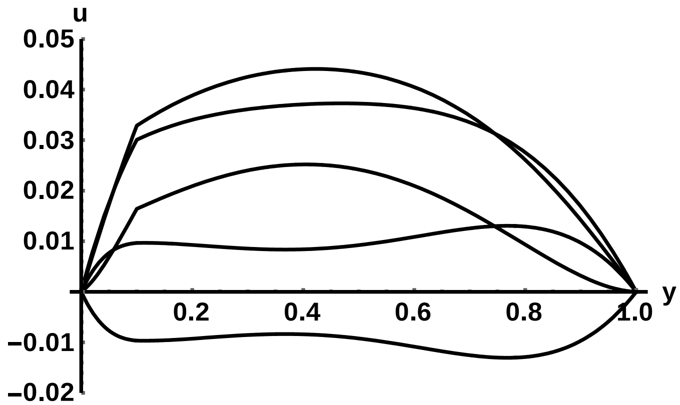


Figure 4. Typical velocity profiles for oscillating pressure gradient in a channel (Figure1(c)) $s = 5, \delta = 0.1, \alpha = \beta = 2$. From top at middle: $ts^2 = 0.5\pi, 0.25\pi, 0.75\pi, 0, \pi$.

I) Longitudinal oscillation of a cylinder

Figure1(d) shows a solid cylinder oscillating longitudinally. There exists a natural length scale *L* which is the radius of the cylinder. The normalized equations, similar to Eqs.(5), are

$$w_{1t} = w_{1rr} + w_{1r} / r, \qquad \beta^2 w_{2t} = w_{2rr} + w_{2r} / r \tag{28}$$

where w is the velocity in the axial direction, and r is the radial coordinate. The matching conditions are

$$w_1 = w_2, \quad \alpha^2 w_{1r} = w_{2r} \tag{29}$$

We distinguish whether the fluid is outside the cylinder or inside the cylinder. If the fluid is outside, let region 2 be the annulus $1 \le r < 1+\delta$ and region 1 be the infinite fluid $1+\delta \le r < \infty$. Let

$$w_1 = e^{is^2 t} f_1(r), \quad w_2 = e^{is^2 t} f_2(r) \tag{30}$$

Eqs. (28) give

$$is^2 f_1 = f_1'' + f_1' / r, \qquad is^2 \beta^2 f_2 = f_2'' + f_2' / r \tag{31}$$

The boundary conditions are

$$\begin{gathered} f_1(\infty) = 0, \quad f_2(1) = 1 \\ f_1(1+\delta) = f_2(1+\delta), \quad \alpha^2 f_1'(1+\delta) = f_2'(1+\delta) \end{gathered} \tag{32}$$

The solution is

$$\begin{gathered} f_1 = c_1 K_0(k_1 r) \\ f_2 = \frac{K_0(k_2 r)}{K_0(k_2)} + c_2 \left[K_0(k_2) I_0(k_2 r) - I_0(k_2) K_0(k_2 r) \right] \end{gathered} \tag{33}$$

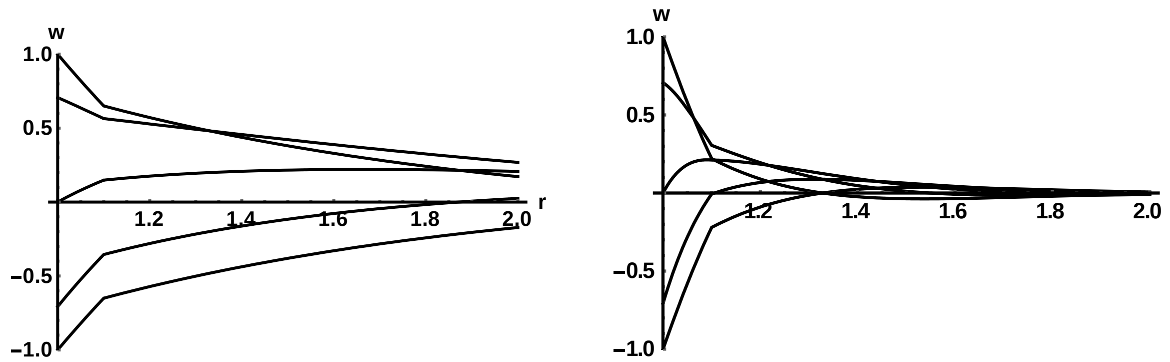
Here k_1, k_2 are defined in Eqs.(16) I_0, K_0 are modified Bessel functions, and

$$\begin{gathered} c_1 = \frac{\beta[I_1(h_2)K_0(h_2) + I_0(h_2)K_1(h_2)]}{K_0(h_1)[\beta I_1(h_2)K_0(h_3) + \alpha^2 I_0(h_2)K_1(h_3)] + I_0(h_1)[\beta K_0(h_3)K_1(h_2) - \alpha^2 K_0(h_2)K_1(h_3)]} \\ c_2 = \frac{[\beta K_0(h_3)K_1(h_2) - \alpha^2 K_0(h_2)K_1(h_3)] / K_0(h_1)}{K_0(h_1)[\beta I_1(h_2)K_0(h_3) + \alpha^2 I_0(h_2)K_1(h_3)] + I_0(h_1)[\beta K_0(h_3)K_1(h_2) - \alpha^2 K_0(h_2)K_1(h_3)]} \end{gathered} \tag{34}$$

where

$$h_1 = \sqrt{i}\beta s, \quad h_2 = \sqrt{i}\beta(1+\delta)s, \quad h_3 = \sqrt{i}(1+\delta)s \tag{35}$$

Figure 5 shows the velocity profiles for longitudinal oscillation outside a cylinder. For higher frequencies, the velocities are more confined.



(a) **(b)**

Figure 5. Typical velocity profiles for longitudinal oscillation of a cylinder in infinite fluid (Figure1(d)) $\delta = 0.1, \alpha = \beta = 2$. From top at left: $ts^2 = 0, 0.25\pi, 0.5\pi, 0.75\pi, \pi$. (a) s =1 (b) s = 5.

For fluid flow inside the cylinder due to longitudinal boundary oscillations, Eq.(31) is the same with the boundary conditions

$$\begin{aligned} &f_2(1) = 1, \quad f_1'(0) = 0 \\ &f_1(1-\delta) = f_2(1-\delta), \quad \alpha^2 f_1'(1-\delta) = f_2'(1-\delta) \end{aligned} \tag{36}$$

The solution is

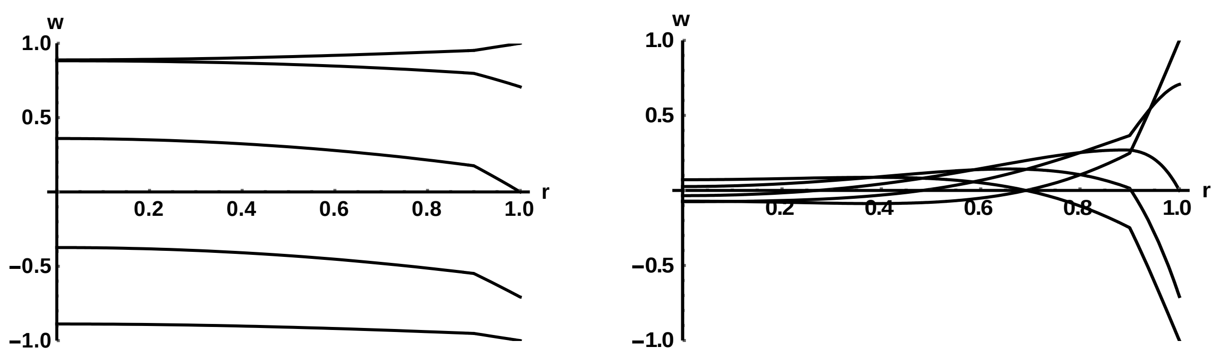
$$f_1 = c_1 I_0(k_1 r) \tag{37}$$

while f_2 has the same form as in Eq.(33). The coefficients are

$$\begin{aligned} c_1 &= \frac{\beta[I_1(h_4)K_0(h_4) + I_0(h_4)K_1(h_4)]}{\alpha^2 I_1(h_5)[I_0(h_1)K_0(h_4) - I_0(h_4)K_0(h_1)] + \beta I_0(h_5)[I_1(h_4)K_0(h_1) + I_0(h_1)K_1(h_4)]} \\ c_2 &= \frac{[\alpha^2 I_1(h_5)K_0(h_4) + \beta I_0(h_5)K_1(h_4)] / K_0(h_1)}{\alpha^2 I_1(h_5)[I_0(h_1)K_0(h_4) - I_0(h_4)K_0(h_1)] + \beta I_0(h_5)[I_1(h_4)K_0(h_1) + I_0(h_1)K_1(h_4)]} \end{aligned} \tag{38}$$

Here

$$h_4 = \sqrt{i}\beta(1-\delta)s, \quad h_5 = \sqrt{i}(1-\delta)s \tag{39}$$



(a) **(b)**

Figure 6. Typical velocity profiles for longitudinal oscillation inside a cylinder (Figure1(d)) $\delta = 0.1, \alpha = \beta = 2$. From top at right: $ts^2 = 0, 0.25\pi, 0.5\pi, 0.75\pi, \pi$. (a) s =1 (b) s =5.

Figure 6 shows at low frequencies the inside fluid almost moves as a whole with the boundary, while at large frequencies a boundary layer is evident.

II) Rotational oscillation of a cylinder

Figure 1(d) shows the solid cylinder performing rotational oscillations about its axis. The normalized governing equations for concentric flow are

$$\begin{aligned} \upsilon_{1t} &= \upsilon_{1rr} + \frac{\upsilon_{1r}}{r} - \frac{\upsilon_1}{r^2} \\ \beta^2 \upsilon_{2t} &= \upsilon_{2rr} + \frac{\upsilon_{2r}}{r} - \frac{\upsilon_2}{r^2} \end{aligned} \tag{40}$$

where υ is the azimuthal velocity. Let

$$\upsilon_1 = e^{is^2 t} f_1(r), \quad \upsilon_2 = e^{is^2 t} f_2(r) \tag{41}$$

Eqs.(40) become

$$\begin{aligned} is^2 f_1 &= f_1'' + f_1'/r - f_1/r^2 \\ is^2 \beta^2 f_2 &= f_2'' + f_2'/r - f_2/r^2 \end{aligned} \tag{42}$$

If the fluid is outside the cylinder, the boundary conditions are

$$f_1(\infty) = 0, \quad f_2(1) = 1$$

The matching conditions are a little different

$$f_1(1+\delta) = f_2(1+\delta), \quad \alpha^2 (f_1' - f_1/r)\big|_{r\to 1+\delta} = (f_2' - f_2/r)\big|_{r\to 1+\delta} \tag{43}$$

The solution is

$$\begin{aligned} f_1 &= c_1 K_1(k_1 r) \\ f_2 &= \frac{K_1(k_2 r)}{K_1(k_2)} + c_2[K_1(k_2) I_1(k_2 r) - I_1(k_2) K_1(k_2 r)] \end{aligned} \tag{44}$$

where

$$c_1 = \frac{\{K_1(h_2)[I_0(h_2) + I_2(h_2)] + I_1(h_2)[K_0(h_2) + K_2(h_2)]\} h_2 / 2}{\{\alpha^2 [I_1(h_2) K_1(h_1) - I_1(h_1) K_1(h_2)][h_3 K_0(h_3) + 2K_1(h_3)] + K_1(h_1) K_1 h_3)[h_2 I_0(h_2) - 2I_1(h_2)] + I_1(h_1)[h_2 K_0(h_2) + 2K_1(h_2)]\}}$$

$$c_2 = \frac{\{K_1(h_3)[h_2 K_0(h_2) + 2K_1(h_2) + (1+\delta) h_1 K_2(h_2)] - \alpha^2 K_1(h_2)[h_3 K_0(h_3) + 2K_1(h_3) + (1+\delta)\sqrt{i} s K_2(h_3)]\} / K_1(h_1)}{\{h_2 K_1(h_1) K_1(h_3)[I_0(h_2) + I_2(h_2)] + 2I_1(h_2) K_1(h_1)[\alpha^2 h_3 K_0(h_3) + (2\alpha^2 - 1) K_1(h_3)] + I_1(h_1) K_1(h_3)[h_2 K_0(h_2) + 2K_1(h_2) + (1+\delta) h_1 K_2(h_2)] - \alpha^2 I_1(h_1) K_1(h_2)[h_3 K_0(h_3) + 2K_1(h_3) + (1+\delta)\sqrt{i} s K_2(h_3)]\}} \tag{45}$$

Figure7 shows the velocity profiles for s=1 and s=5.

As frequency approaches zero, our oscillatory solution agrees with the steady state solution:

$$\upsilon_1 = \frac{(1+\delta)^2}{[1+\alpha^2 \delta(2+\delta)] r}, \quad \upsilon_2 = r - \frac{\alpha^2 (1+\delta)^2}{[1+\alpha^2 \delta(2+\delta)]} \left(r - \frac{1}{r} \right) \tag{46}$$

If the fluid is inside the cylinder, Eqs.(42) still hold but with the boundary conditions

$$f_2(1)=1,\quad f_1(0)=0$$
$$f_1(1-\delta)=f_2(1-\delta),\quad \alpha^2\left(f_1'-\frac{f_1}{r}\right)\Big|_{r=1-\delta}=\left(f_2'-\frac{f_2}{r}\right)\Big|_{r=1-\delta} \tag{47}$$

The solution is

$$f_1=c_1I_1(k_1r)$$
$$f_2=\frac{K_1(k_2r)}{K_1(k_2)}+c_2[K_1(k_2)I_1(k_2r)-I_1(k_2)K_1(k_2r)] \tag{48}$$

where

$$c_1=\frac{h_4\{K_1(h_4)[I_0(h_4)+I_2(h_4)]+I_1(h_4)[K_0(h_4)+K_2(h_4)]\}}{\{\alpha^2[-2I_1(h_5)+h_5[I_0(h_5)+I_2(h_5)]][I_1(h_1)K_1(h_4)-I_1(h_1)K_1(h_4)]+ \\ I_1(h_5)K_1(h_1)[2h_4I_0(h_4)-4I_1(h_4)]+I_1(h_1)[2h_4K_0(h_4)+4K_1(h_4)\ ]\ ]}$$

$$c_2=\frac{\{h_5K_1(h_4)[I_0(h_5)+I_2(h_5)]+2I_1(h_5)[h_4K_0(h_4)-(\alpha^2-2)K_1(h_4)]\}/K_1(h_1)}{\{K_1(h_1)[h_4I_0(h_4)I_1(h_5)-\alpha^2h_5I_0(h_5)I_1(h_4)-2(1-\alpha^2)I_1(h_4)I_1(h_5)+ \\ h_4I_1(h_5)I_2(h_4)-\alpha^2h_5I_1(h_4)I_2(h_5)]+I_1(h_1)\langle\alpha^2h_5K_1(h_4)[I_0(h_5)+I_2(h_5)]+ \\ I_1(h_5)[2h_4K_0(h_4)-2(\alpha^2-2)K_1(h_4)\rangle\}} \tag{49}$$

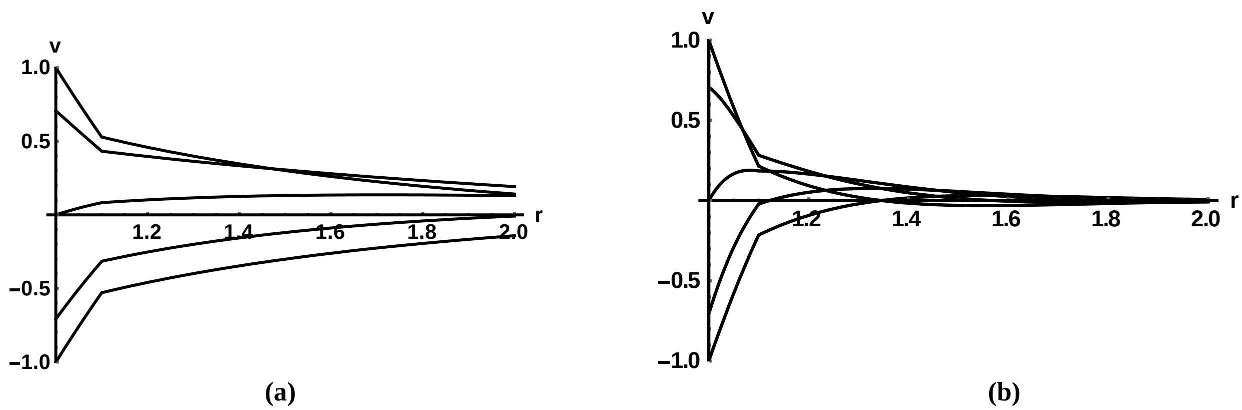


Figure 7. Typical velocity profiles for rotational oscillation of a cylinder in infinite fluid (Figure1(e)) $\delta=0.1$, $\alpha=\beta=2$. From top at left: $ts^2=0, 0.25\pi, 0.5\pi, 0.75\pi, \pi$. (a) s =1 (b) s =5.

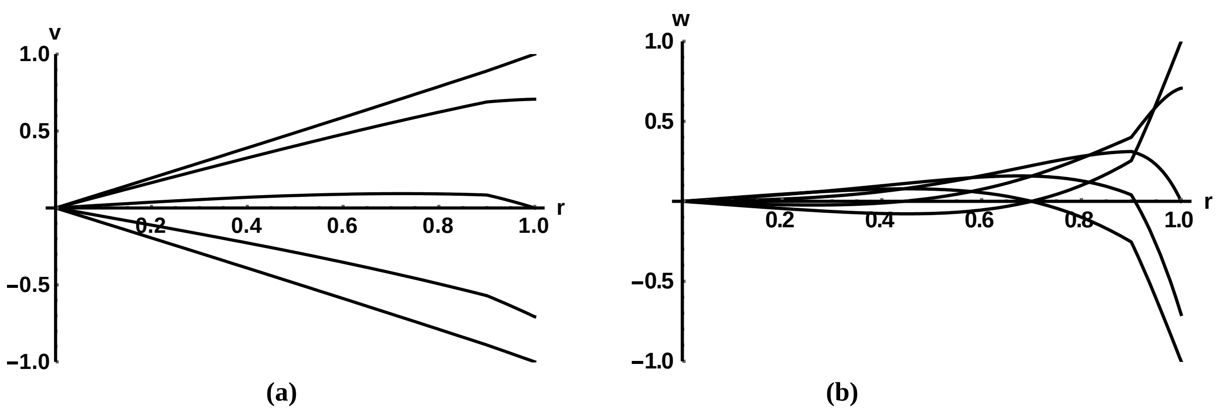


Figure 8. Typical velocity profiles for rotational oscillation inside a cylinder (Figure1(e)) $\delta=0.1$, $\alpha=\beta=2$. From top at right: $ts^2=0, 0.25\pi, 0.5\pi, 0.75\pi, \pi$. (a) s =1, (b) s =5.

Figure8 shows typical velocity profiles for *s*=1 and *s*=5. In the steady state, the interior would just be rigid rotation.

III) Oscillatory pressure inside a cylinder

This case is especially relevant to pulsatile flow in small blood vessels (Figure1 (f)). Bugliarello and Sevilla [14] formulated the problem but did not give the closed form solution or any results.

Using Eqs,(1, 30) the normalized governing equations similar to Eq.(8) are

$$is^2 f_1 = 1 + f_1'' + f_1'/r, \quad is^2\beta^2 f_2 = \alpha^2 + f_2'' + f_2'/r \tag{50}$$

The boundary conditions are Eqs.(36). The solution is

$$f_1 = \frac{1}{is^2}\left[1 + c_1 I_0(k_1 r)\right]$$

$$f_2 = \frac{\alpha^2}{is^2\beta^2}\left\{\left[1 - \frac{K_0(k_2 r)}{K_0(k_2)}\right] + c_2\left[K_0(k_2)I_0(k_2 r) - I_0(k_2)K_0(k_2 r)\right]\right\} \tag{51}$$

where

$$c_1 = \frac{\{(\alpha^2-\beta^2)[I_1(h_4)K_0(h_1)+I_0(h_1)K_1(h_4)] - \alpha^2[I_1(h_4)K_0(h_4)+I_0(h_4)K_1(h_4)]\}/\beta}{\alpha^2 I_1(h_5)[I_0(h_1)K_0(h_4)-I_0(h_4)K_0(h_1)] + \beta I_0(h_5)[I_1(h_4)K_0(h_1)+I_0(h_1)K_1(h_4)]}$$

$$c_2 = \frac{\{I_1(h_5)[(\beta^2-\alpha^2)K_0(h_1)+\alpha^2 K_0(h_4)] + \beta I_0(h_5)K_1(h_4)\}/K_0(h_1)}{\alpha^2 I_1(h_5)[I_0(h_4)K_0(h_1)-I_0(h_1)K_0(h_4)] - \beta I_0(h_5)[I_1(h_4)K_0(h_1)+I_0(h_1)K_1(h_4)]} \tag{52}$$

Figure 9 shows typical velocity distributions for *s*=1 and *s*=5. Notice at low frequencies the velocity is almost parabolic and for high frequencies the maximum velocity moved to the sides. Thus "Richardson's annular effect" [18] exists also for two-fluid oscillatory flow.

In the limit of $s \to 0$ our oscillatory solution agrees with the steady state solution which is composed of two paraboloids

$$w_1 = \frac{-r^2}{4} + \frac{1}{4}\left[(1-\delta)^2 + \alpha^2\delta(2-\delta)\right], \quad w_2 = \frac{\alpha^2}{4}(1-r^2) \tag{53}$$

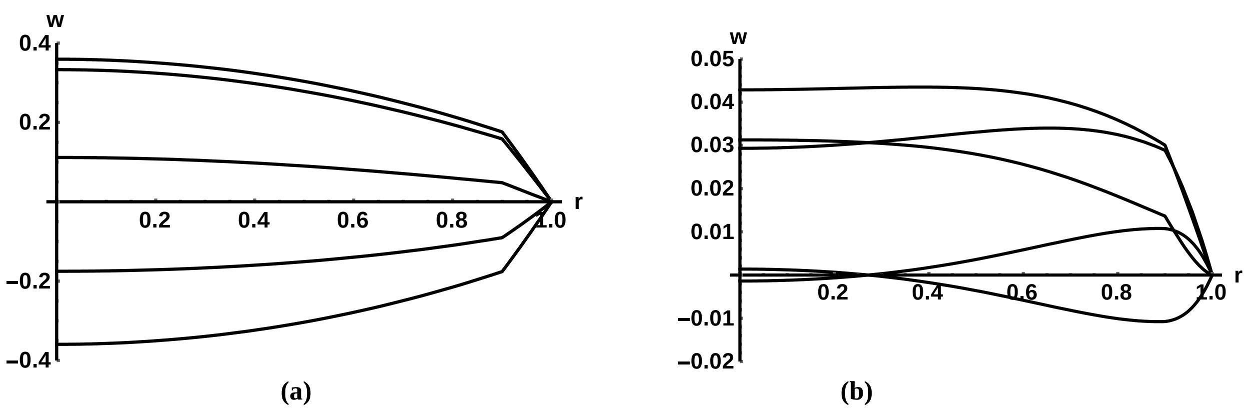


Figure 9. Typical velocity profiles inside a cylinder due to oscillation of the pressure gradient (Figure 1(f)) $\delta = 0.1$, $\alpha = \beta = 2$. From top at left: $ts^2 = 0, 0.25\pi, 0.5\pi, 0.75\pi, \pi$. (a) *s* =1 (b) *s* =5.

5. Testing Navier's partial slip boundary condition

Almost two hundred years ago Navier [16], unsure of the no slip boundary condition, suggested a partial slip condition for flow over a solid surface. Navier stated the apparent slip velocity is proportional to the local shear stress

$$u' = S'\tau' \tag{54}$$

For parallel flow in normalized form Eq.(54) is

$$u = S\frac{\partial u}{\partial n}, \qquad S = S'\rho\nu / L \tag{55}$$

where *n* is the normal coordinate and *S* is the normalized slip coefficient, or the slip length. Navier's condition can be shown approximate, theoretically and experimentally, *steady* flows of rarefied gases, flows with lubricated boundary, and flow over rough surfaces.

A search in the Science Citation Index shows that there exist dozens of reports applying Navier's condition to *unsteady* flow problems. But is Navier's condition valid for unsteady flows?

For viscous flow over a lubricated boundary, there exists a thin fluid boundary layer separating a bulk flow of higher viscosity. Although the no-slip condition is still applied to the lubricant, the bulk flow experiences apparent partial slip.

For two-fluid starting flow Wang [12] showed Navier's condition fails for the core fluid. Wang [11] solved the oscillatory symmetric two-fluid flow in a channel, and the core fluid also does not reflect Navier's condition. These unsteady examples are all two-dimensional and all caused by unsteady pressure gradients, i.e. Poiseuille type flows.

This paper presents oscillatory solutions for two-fluid flows. Since these solutions are exact and closed-form, they are particularly suited for testing Navier's condition.

Consider first the Couette type two-fluid flow between two plates caused by a bottom oscillatory plate. Let $\delta = 0.9,\ \alpha = \beta = 0.3$ such that the lower (heavier) fluid be the bulk fluid and the top layer be the lubricating fluid. If Navier's condition is valid, the normalized slip length *S* in Eq. (54) should remain constant. For the steady state solution Eq. (21), we find

$$S = \frac{u_2}{-u_{2y}}\Big|_{y\to\delta} = \frac{1-\delta}{\alpha^2} = 1.11 \tag{56}$$

which is indeed a constant. Figure 10(a) shows the ratio *S* for low frequency oscillation (*s*=0.5) is close to the steady-state value but becomes infinite at certain times. It is questionable whether these infinite spikes would affect the flow if Navier's condition is applied. Figure 10(b) shows at higher frequencies (*s*=5) the ratio *S* varies greatly with time. In particular *S* becomes infinite at the spikes. The reason can be explained as follows. The velocity profiles for this case, shown in Figure 3, are enlarged in Figure10(c) near the time of a spike. It is seen that while the velocity is still positive, the shear stress changed sign since the kinks at the interface at $y = \delta$ changed sign. Thus near *t*=0.086 the slip length becomes infinite.

Next consider the two-fluid flow caused by oscillatory pressure in a channel. The velocity profiles are shown in Figure 4 for $\delta = .1,\ \alpha = \beta = 2$ where the upper fluid is the bulk and the lower fluid has less viscosity (e.g. oil over water). The ratio *S* is shown in Figure11 where we see spikes and non-constant ratio *S*. Similar results are shown in Figure 12, for oscillatory pressure-induced two-fluid tube flow. In fact all the exact solutions in this paper do not support Navier's condition.

Together with the previous reports, we conclude that, in general, Navier's condition cannot be applied to unsteady lubricated flows.

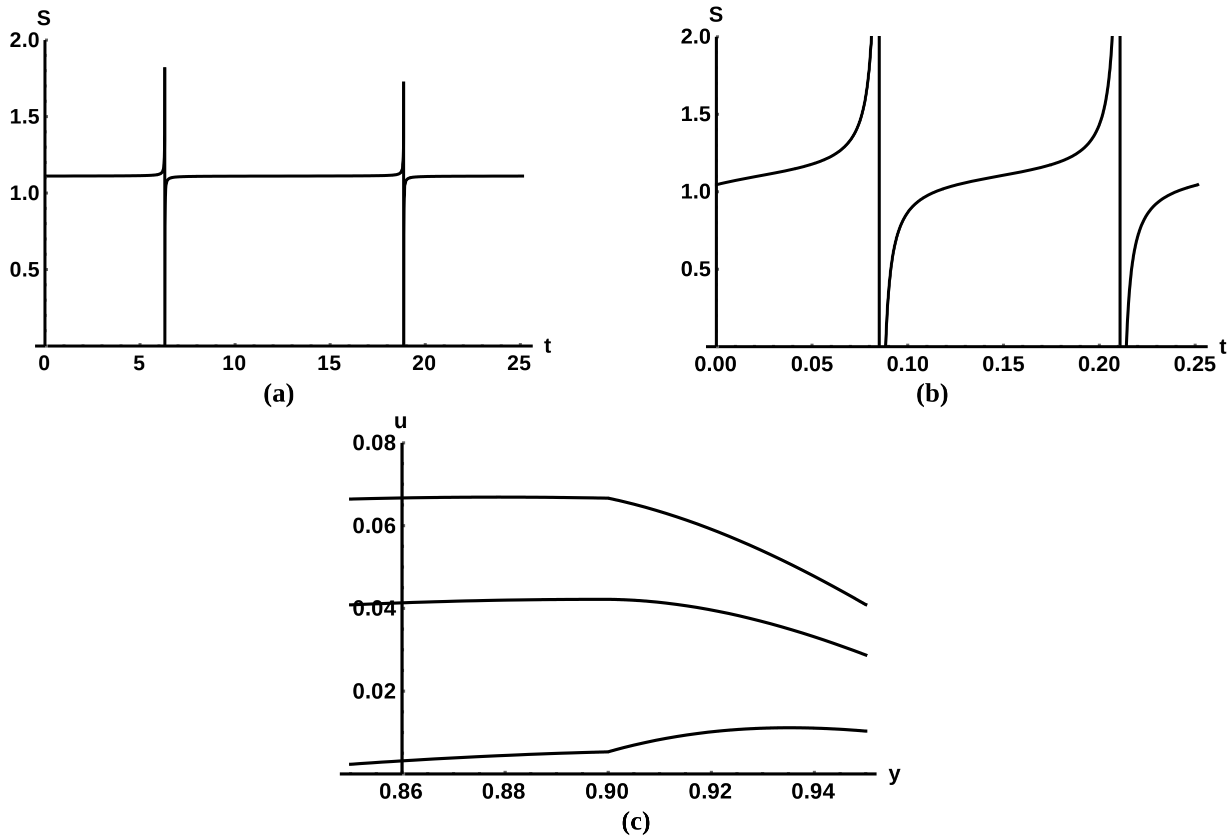


Figure 10. Normalized slip length *S* for flow in a channel with one plate oscillating $\delta = 0.9, \alpha = \beta = 0.3$. (a) *s* =0.5. (b) *s* =5. (c) Detailed velocity profile of Figure 3 From top: *t*=0.083, 0.086, 0.088.

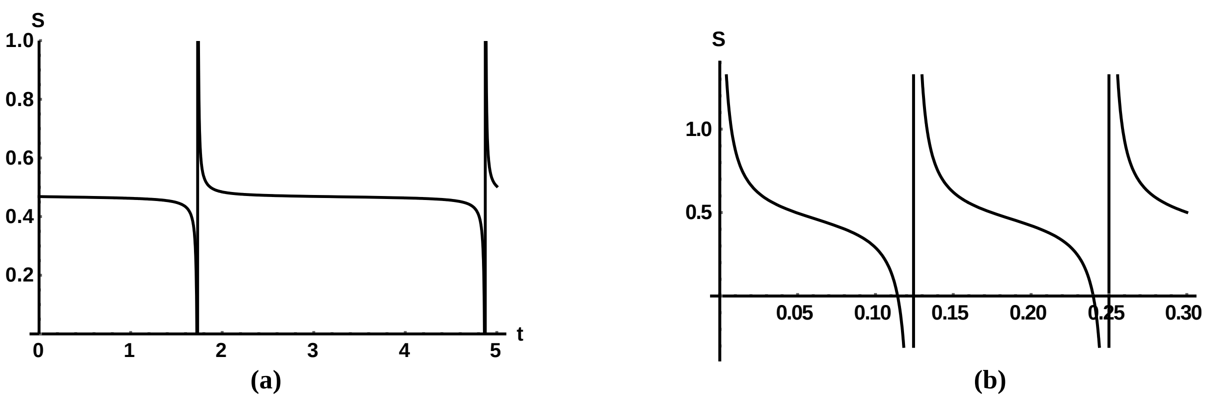


Figure 11. Normalized slip length *S* for flow in a channel due to oscillating pressure gradient $\delta = 0.1, \alpha = \beta = 2$ (a) *s* =1 (b) *s* =5.

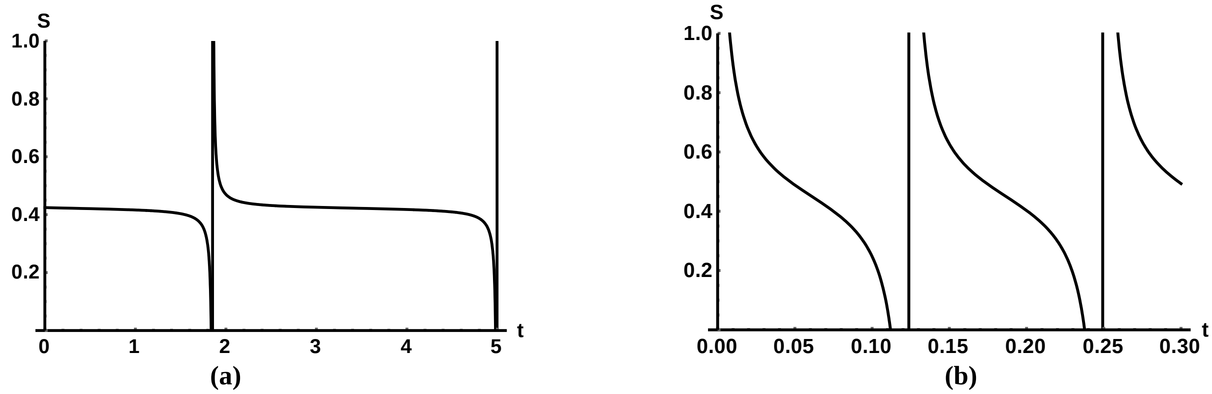


Figure 12. Normalized slip length *S* for flow in a tube due to oscillating pressure gradient $\delta = 0.1, \alpha = \beta = 2$ (a) *s* =1 (b) *s* =5.

6. Discussions and Conclusions

Table 1 shows a summary of the exact solutions found. All of these solutions are new and archival.

Table 1 Exact two-fluid oscillatory solutions.

Description	Figure	Solution
In-plane oscillation of a single plate (Couette)	Fig.1(a)	Eqs.(12,14-17)
Channel with one in-plane oscillating wall. (Couette)	Figure1(b)	Eqs.(12,15,18-20)
Chanel with oscillating pressure gradient (Poiseuille)	Figure1(c)	Eqs.(12,25,26)
Longitidinal oscillation of a circular cylinder with fluid outside (Couette)	Figure1(d)	Eqs.(30,33-35)
Longitidinal oscillation of a circular cylinder with fluid intside (Couette)	Fig.1(d)	Eqs.(30,33,37-39)
Rotational oscillation of a circular cylinder with fluid outside (Couette)	Fig.1(e)	Eqs.(41,44,45)
Rotationnal oscillation of a circular cylinder with fluid inside (Couette)	Fig.1(e)	Eqs.(41,48,49)
Pressure oscillation inside a circular cylinder (Poiseuille)	Fig.1(f)	Eqs.(30,51,52)

As we noted previously, closed-form solutions serve as benchmarks for all approximate methods, including numerical and series solutions. This paper considers some basic oscillatory two-fluid flows. Extensions are possible, such as stratified flows with several layers and concentric annular flows. Starting flows and other unsteady flows may be solved by Laplace transform, but could not be in closed form.

Aside from geometric factors, oscillatory flow is governed by the parameter s in Eq.(9). Some estimates of the magnitude of s are as follows. For normal blood flow in arterioles of 100 micron diameter, and a resting frequency of 60 beats per minute, we find s is about 0.1. This estimate would be increased by an order in exertion. For the transport of olive oil which has been wetted by a layer of water, the density ratio γ is about unity and the kinematic viscosity of olive oil is 30 times that of water. For the steady-state one can show the resistance through the tube is greatly reduced by the water layer. For a tube of 2 cm radius and a reciprocating pump of 1 cycle per second, the value of s is about 9. The range of s can be large for oscillations caused by external vibrations.

Any periodic oscillation can be Fourier-decomposed into individual harmonics. Since the problems are linear, our single frequency results can be superposed appropriately. For the higher harmonics the frequency would be integer multiples of the basic frequency.

Using our exact closed-form solutions, together with existing evidence, we find Navier's partial slip condition cannot be applied to unsteady lubricating flows. The reason is due to the two fluids do not oscillate in phase, causing a mismatch in velocity and shear stress at the interface. Thus Navier's condition fails.

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